27.03.2017, 17:00-18:30

MATH1002/MATH102 Calculus II Midterm Exam

Name Surname: ______________________________ Student Number: ___________

Department: _________________________________ Signature: ________________

In solving the following problems, you are required to show all your work and provide the necessary explanations everywhere to get full credit. Each problem is 18 points.

Problem 1: [18 points] Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)}$$

Problem 2: [18 points] Find the center and the radius of convergence of the following series. Does the series converge at the endpoints of its interval of convergence?

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 5^n}$$

Problem 3: [18 points] Consider the points **A(1, 1 0) B(2, 0,1) C(0, 1, 3).**

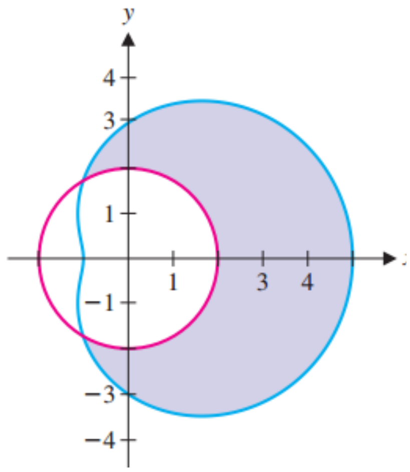
a. Find the scalar and vectoral products: $(\overrightarrow{AB} \cdot \overrightarrow{AC})$ and $(\overrightarrow{AB} \times \overrightarrow{AC})$
b. Find an equation for the plane that contains A, B and C
c. Find the parametric equation of the line that contains A and C

Problem 4: [18 points] Find the area inside the limaçon $r = 3 + 2\cos\theta$ and outside the circle $r = 2$.

Hint:

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$



Problem 5: [18 points] Find the tangent line to the curve ***x = 2cost, y = 2sint*** at ***t=π/4***. Also find the value of d^2y/dx^2 at this point.

Problem 6: [18 points] Write down the Taylor series at $x = -1$ for the function $f(x) = \frac{1}{x^2}$.

(1) $\dfrac{1}{(2n+1)(2n-1)} = \dfrac{A}{2n+1} + \dfrac{B}{2n-1} \Rightarrow A(2n-1) + B(2n+1) = 1$

$n = \frac{1}{2} \Rightarrow \quad B = \frac{1}{2}$

$n = -\frac{1}{2} \Rightarrow \quad A = -\frac{1}{2}.$

$$\sum_{n=1}^{N} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{2N-1}} - \frac{1}{2N+1}\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{2N+1}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} = \lim_{N\to\infty} \frac{1}{2}\left(1 - \frac{1}{2N+1}\right) = \frac{1}{2}.$$

(2) $\displaystyle \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{|x-2|^n} = \lim_{n\to\infty} |x-2| \left(\frac{n}{n+1}\right)^2 \cdot \frac{1}{5} = \frac{|x-2|}{5}$

By ratio test the series converge if $|x-2| < 5$.

Radius of convergence $= 5$, Center of convergence $= 2$.

if $x-2 = 5 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p=2$ series)

if $x-2 = -5 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by Alternating Series Test ($a_n = \frac{1}{n^2}$ is positive, decreasing, $\lim_{n\to\infty} \frac{1}{n^2} = 0$)

(3) $\vec{AB} = \langle 1,-1,1\rangle, \quad \vec{AC} = \langle -1,0,3\rangle.$

a) $\vec{AB}\cdot\vec{AC} = -1+0+3 = 2.$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ -1 & 0 & 3 \end{vmatrix} = i(-3-0) - j(3+1) + k(0-1)$$

$$= -3i - 4j - k.$$

b) $\vec{n} = \vec{AB}\times\vec{AC}$

$P(x,y,z)$, $\vec{AP} = \langle x-1, y+1, z-1\rangle$

$$0 = \vec{n}\cdot\vec{AP} = -3(x-1) - 4(y+1) - (z-1) = 0$$

$$-3x - 4y - z = 0$$

$$3x+4y+z=0.$$

c) $\vec{r}(t) = \vec{OA} + t\vec{AC} = \langle 1,1,0\rangle + t\langle -1,0,3\rangle$

$$= \langle 1-t, 1, 3t\rangle$$

(4) $3+2\cos\theta = 2 \Rightarrow 2\cos\theta = -1 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \mp\frac{2\pi}{3}$ $(\mp 120°)$

$$A = \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} \left[(3+2\cos\theta)^2 - 2^2\right] = \frac{1}{2}\int_{-\frac{2\pi}{3}}^{2\pi/3} \left[5+12\cos\theta+4\cos^2\theta\right]d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= \frac{1}{2}\int_{-\frac{2\pi}{3}}^{2\pi/3}\left[5+12\cos\theta+4\,\frac{(\cos 2\theta+1)}{2}\right]d\theta$$

$$= \frac{1}{2}\left[7\theta + 12\sin\theta + \sin 2\theta\right]_{\theta=-2\pi/3}^{2\pi/3} = \frac{1}{2}\left[7\cdot 2\cdot\frac{2\pi}{3} + 12\cdot 2\cdot\frac{\sqrt{3}}{2} + 2\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{14\pi}{3} + 11\frac{\sqrt{3}}{2}$$

(5) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t}$ $\quad t=\frac{\pi}{4} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -1.$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'/dt}{dx/dt} = \frac{\frac{d}{dt}\left(-\frac{\cos t}{\sin t}\right)}{-2\sin t} = \frac{\frac{1}{\sin^2 t}}{-2\sin t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/4} = \frac{-1}{2\,\frac{2^{3/2}}{2^3}} = \frac{-4}{2^{3/2}} = -\sqrt{2}$$

(6) $f(x) = \frac{1}{x^2} \Rightarrow f(-1) = 1$

$f'(x) = \frac{-2}{x^3} \Rightarrow f'(-1) = 2$

$f''(x) = \frac{6}{x^4} \Rightarrow f''(-1) = 6$

$f'''(x) = \frac{-24}{x^5} \Rightarrow f'''(-1) = 24.$

$\vdots$

$f^{(n)}(-1) = (n+1)!$

Taylor series at $x=-1$,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n = \sum_{n=0}^{\infty} (n+1)(x+1)^n.$$